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Exponential stabilization of an axially moving string with geometrical nonlinearity by linear boundary feedback

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Abstract

This paper addresses the vibration suppression problem of an axially moving string using boundary control. Taking into account large transverse motion of the string via the second-order component of Lagrangian strain, a nonlinear oscillation equation is derived for the problem by employing the Hamilton's Principle. Direct Lyapunov method is then adopted to demonstrate that a linear boundary control of negative speed feedback ensures the vibration response of the moving string exponentially stable. The main contributions are a new stability condition in terms of controller gain parameter, and a detailed demonstration on the equilibrium condition at the boundary where the control is applied.

1. Introduction

Axially moving string is a typical model widely used to represent threads, wires, magnetic tape, belts, band-saws and cables, especially when the subject concerned is long and narrow enough, and has a negligible flexural rigidity. Various mathematical models and approaches have been established for a better understanding with linear or nonlinear dynamic behavior of these moving continua [1–7].

In order to suppress the transverse or longitudinal vibration of an axially moving string, researches into controllability of the system from different engineering background have also been conducted, especially using boundary control. Most of the controllers, including passive control law with linear damping, linear discrete system (mass–damping–spring) or various active control laws, are designed on the basis of linear models of axially moving strings [8–15].

To the best of the authors' knowledge, there are very little publications which employ nonlinear models of axially moving strings for vibration control. In Refs. [16,17], two nonlinear models were used to reflect the dynamic behavior of axially moving string, and a passive linear damping controller was adopted for vibration reduction. Theoretical derivation via direct Lyapunov method revealed the possible values of the gain parameter to ensure vibration responses of both nonlinear dynamic models asymptotically stable.

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Using Hamilton's Principle and incorporating the second-order component of the Lagrangian strain, a nonlinear model about transverse oscillation is derived in this paper for a moving string which runs across two eyelets with a constant speed and is subjected to certain boundary control. The equation is almost similar to that adopted in Ref. [16], but the boundary condition is of more generality and has more straightforward physical meanings before being changed to an identical expression as in Ref. [16]. To this new model, a series of Lyapunov-type energy functionals are constructed. After proving some theorems and corollaries, it is demonstrated that the oscillation of the nonlinear moving string can be suppressed by a linear boundary control which is a negative feedback of the speed of the string at the movable end. A refined domain of the controller gain parameter is obtained in which the vibration response of the moving string is exponentially stable. One of the corollaries is identical to the main theorem obtained in Ref. [16].

2. Equation of motion

In Fig. 1, a string is pulled at a constant speed v through two eyelets. The eyelet at $x = 0$ is fixed and the one at $x = l$ can move freely in the direction of the Y-axis. A control input force $F_c(t)$ for $t \geqslant 0$ is applied to the free-to-move eyelet in the direction of Y.

Let t be the time, x be the spatial coordinate along the before-deformed string, $y(x,t)$ be the transversal displacement of the string at spatial coordinate x and time t, and ρ be the mass per unit length.

Because the string travels with a constant speed v, the longitudinal and transverse velocity components, measured by a stationary observer, are v and $y_t + vy_x$, respectively, where the subscript notations denote partial differential, i.e., $(\cdot)_t = \partial(\cdot)/\partial t$, $(\cdot)_x = \partial(\cdot)/\partial x$.

Hamilton's Principle

$$\int_{t_0}^{t_1} (\delta T - \delta U + \delta W)\,\mathrm{d}t = 0 \tag{1}$$

is applied to obtain the governing equation and boundary conditions of the system. Here T, U, δW are, respectively, the kinetic energy, the potential energy between $x = 0$ and l, and the virtual work done by the external force, which can be calculated according to the following expressions as:

$$T = \frac{1}{2}\int_0^l \rho\left[v^2 + \left(y_t + vy_x\right)^2\right]\mathrm{d}x, \quad U = \frac{1}{2}\int_0^l \left(T_0 + \frac{EA}{4}y_x^2\right)y_x^2\,\mathrm{d}x, \quad \delta W = F_c(t)\delta y(l,0). \tag{2}$$

Here T_0, E and A are separately the initial tension, the Young's modulus, and the cross-sectional area of the string. It is noted that the expression of potential energy has been obtained by incorporating the second-order Lagrangian strain, $y_x^2/2$, which introduces the geometrical nonlinearity of the string.

Integrating Eq. (1) by parts with $\delta y(x, t_0) = \delta y(x, t_1) = 0$ and $\delta y(0, t) = 0$, leads to the governing equation

$$\rho\left(y_{tt} + vy_{xt}\right) + \rho v\left(y_{tx} + vy_{xx}\right) - \left[T_0 + \frac{3EA}{2}y_x^2\right]y_{xx} = 0 \tag{3}$$

and the boundary condition at $x = l$:

$$\rho v\left(y_t + vy_x\right) - \left[T_0 + \frac{EA}{2}y_x^2\right]y_x + F_c(t) = 0. \tag{4}$$

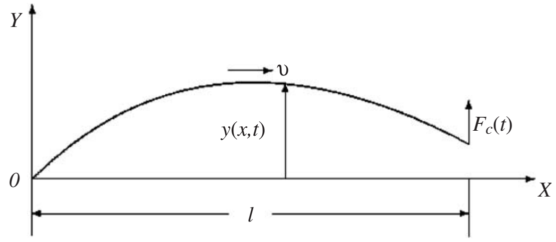

Fig. 1. Schematic model of a traveling string with boundary control.

Introducing following non-dimensional variables:

$$y^* = \frac{y}{l}, \quad x^* = \frac{x}{l}, \quad t^* = \frac{t}{l}\sqrt{\frac{T_0}{\rho}}, \quad v^* = v\sqrt{\frac{\rho}{T_0}}, \quad b = \frac{EA}{T_0}, \quad F_c^*(t^*) = \frac{F_c(t)}{T_0}, \tag{5}$$

inserting them into Eqs. (3) and (4), and dropping all stars from now on, yield

$$y_{tt} + 2vy_{xt} = \left[1 - v^2 + \frac{3}{2}by_x^2\right]y_{xx} \quad \text{for } x \in (0,1) \tag{6}$$

and

$$\left[1 - v^2 + \frac{1}{2}by_x^2\right]y_x - vy_t = F_c(t) \quad \text{for } x = 1, \tag{7}$$

respectively. Substituting av for v in above equations and hence letting $u(t) = F_c(t) + vy_t(1,t)$ result in the equation of motion and boundary condition adopted in Ref. [16].

3. Stabilization by boundary control

In this section, the stabilization of solutions of the model derived above for a moving nonlinear string is analyzed. For the sake of easy reading and later referring, the governing equation, the boundary conditions and the initial functions are put together as

$$y_{tt}(x,t) + 2vy_{xt}(x,t) = \left[1 - v^2 + \frac{3}{2}by_x^2(x,t)\right]y_{xx}(x,t), \quad t>0, 1>x>0, \tag{8a}$$

$$u(t) = \left[1 - v^2 + \frac{1}{2}by_x^2(1,t)\right]y_x(1,t), \quad t \geqslant 0, \tag{8b}$$

$$y(0,t) = 0, \quad y_t(0,t) = 0, \quad t \geqslant 0, \tag{8c}$$

$$y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad 1 \geqslant x \geqslant 0. \tag{8d,e}$$

Here $f(x)$ and $g(x)$ are the initial displacement and velocity of the string, respectively. Function $u(t)$ is the new control input force defined by $u(t) = F_c(t) + vy_t(1,t)$. From derivation in Section 2, it can be observed that both b and v are constant real numbers with $b>0$ and $v \in (0,1)$.

Define a scalar-valued function $V(t)$ as

$$V(t) := E(t) + \gamma \int_0^1 xy_x(x,t)\left[y_t(x,t) + vy_x(x,t)\right] \mathrm{d}x \tag{9}$$

for all $t \geqslant 0$. Here γ is a constant real number, and

$$E(t) := \frac{1}{2}\int_0^1 \left[y_t^2(x,t) + \left(1 - v^2\right)y_x^2(x,t) + \frac{b}{4}y_x^4(x,t)\right] \mathrm{d}x. \tag{10}$$

Proposition 1. *Let γ satisfy*

$$0<\gamma<1 - v. \tag{11}$$

Then, the function $V(t)$ satisfies

$$0 \leqslant K_1^* E(t) \leqslant V(t) \leqslant K_2^* E(t) \tag{12}$$

for all $t \geqslant 0$, where $K_1^*>0$ and $K_2^*>0$ are constant real numbers, given by

$$K_1^* = 1 - \frac{\gamma}{1 - v}, \quad K_2^* = 1 + \frac{\gamma}{1 - v}. \tag{13}$$

Proof.

$$\begin{aligned} V(t) &\leqslant E(t) + \gamma \int_0^1 \left[\left| y_x(x,t) y_t(x,t) \right| + v y_x^2(x,t) \right] \mathrm{d}x \\ &\leqslant E(t) + \gamma \int_0^1 \left[\frac{1}{2(1-v)} y_t^2(x,t) + \frac{1-v}{2} y_x^2(x,t) + v y_x^2(x,t) \right] \mathrm{d}x \\ &= E(t) + \frac{\gamma}{2(1-v)} \int_0^1 \left[y_t^2(x,t) + \left(1 - v^2\right) y_x^2(x,t) \right] \mathrm{d}x \\ &\leqslant E(t) + \frac{\gamma}{(1-v)} E(t), \end{aligned}$$

similarly,

$$\begin{aligned} V(t) &\geqslant E(t) - \gamma \int_0^1 \left[\left| y_x(x,t) y_t(x,t) \right| + v y_x^2(x,t) \right] \mathrm{d}x \\ &\geqslant E(t) - \gamma \int_0^1 \left[\frac{1}{2(1-v)} y_t^2(x,t) + \frac{1-v}{2} y_x^2(x,t) + v y_x^2(x,t) \right] \mathrm{d}x \\ &= E(t) - \frac{\gamma}{2(1-v)} \int_0^1 \left[y_t^2(x,t) + \left(1 - v^2\right) y_x^2(x,t) \right] \mathrm{d}x \\ &\geqslant E(t) - \frac{\gamma}{(1-v)} E(t). \end{aligned}$$

The proposition is thus proved. □

Lemma 1. *The time derivatives of the functions $E(t)$ and $V(t)$ in Eqs.* (10) and (9), *along system* (8) *satisfy*

$$\dot{E}(t) = -v y_t^2(1,t) + (1 - v^2) y_t(1,t) y_x(1,t) + \frac{b}{2} y_t(1,t) y_x^3(1,t), \tag{14}$$

$$\dot{V}(t) = -\frac{\gamma}{2} \int_0^1 \left[y_t^2(x,t) + (1 - v^2) y_x^2(x,t) + \frac{3}{4} b y_x^4(x,t) \right] \mathrm{d}x + F(t) \tag{15}$$

for all $t \geqslant 0$, where

$$\begin{aligned} F(t) := \left(\frac{1}{2}\gamma - v \right) y_t^2(1,t) + \left(1 - v^2\right) y_t(1,t) y_x(1,t) + \frac{b}{2} y_t(1,t) y_x^3(1,t) \\ + \frac{1 - v^2}{2} \gamma y_x^2(1,t) + \frac{3}{8} b \gamma y_x^4(1,t). \end{aligned} \tag{16}$$

Proof. Differentiating Eq. (10) with respect to t yields

$$\dot{E}(t) = \int_0^1 \left[y_t(x,t) y_{tt}(x,t) + (1 - v^2) y_x(x,t) y_{xt}(x,t) + \frac{b}{2} y_x^3(x,t) y_{xt}(x,t) \right] \mathrm{d}x, \tag{17}$$

hence substituting $y_{tt}(x,t)$ from Eq. (8a) into Eq. (17) leads to

$$\begin{aligned} \dot{E}(t) = \int_0^1 \Bigg[y_t(x,t) \left(-2v y_{xt}(x,t) + (1 - v^2 + \frac{3}{2} b y_x^2(x,t)) y_{xx}(x,t) \right) \\ + (1 - v^2) y_x(x,t) y_{xt}(x,t) + \frac{b}{2} y_x^3(x,t) y_{xt}(x,t) \Bigg] \mathrm{d}x. \end{aligned} \tag{18}$$

Using

$$
\begin{aligned}
& y_t(x,t)\Big[-2vy_{xt}(x,t)+(1-v^2+\frac{3}{2}by_x^2(x,t))y_{xx}(x,t)\Big] \\
& \quad +\left(1-v^2\right)y_x(x,t)y_{xt}(x,t)+\frac{b}{2}y_x^3(x,t)y_{xt}(x,t) \\
& = -2vy_t(x,t)y_{xt}(x,t)+(1-v^2)y_t(x,t)y_{xx}(x,t)+\frac{3}{2}by_t(x,t)y_{xx}(x,t)y_x^2(x,t) \\
& \quad +\left(1-v^2\right)y_x(x,t)y_{xt}(x,t)+\frac{b}{2}y_x^3(x,t)y_{xt}(x,t) \\
& = -v\left(y_t^2(x,t)\right)_x+\left(1-v^2\right)\left(y_t(x,t)y_x(x,t)\right)_x+\frac{b}{2}\left(y_t(x,t)y_x^3(x,t)\right)_x
\end{aligned}
$$

and integrating the right-hand side of Eq. (18) with Eq. (8c), Eq. (14) is obtained.

Differentiating $V(t)-E(t)$ results in

$$
\dot{V}(t)-\dot{E}(t)=\gamma\int_0^1 x\left[y_t(x,t)y_{xt}(x,t)+y_{tt}(x,t)y_x(x,t)+2vy_x(x,t)y_{xt}(x,t)\right]\mathrm{d}x. \tag{19}
$$

Subsequently substituting $y_{tt}(x,t)$ from Eq. (8a) into Eq. (19) leads to

$$
\begin{aligned}
\dot{V}(t)-\dot{E}(t)=\gamma\int_0^1 x\Big[& y_t(x,t)y_{xt}(x,t) \\
& +\left(-2vy_{xt}(x,t)+(1-v^2+\frac{3}{2}by_x^2(x,t))y_{xx}(x,t)\right)y_x(x,t) \\
& +2vy_x(x,t)y_{xt}(x,t)\Big]\,\mathrm{d}x.
\end{aligned} \tag{20}
$$

Using

$$
\begin{aligned}
& x\Big[y_t(x,t)y_{xt}(x,t)+\left(-2vy_{xt}(x,t)+(1-v^2+\frac{3}{2}by_x^2(x,t))y_{xx}(x,t)\right)y_x(x,t) \\
& \quad +2vy_x(x,t)y_{xt}(x,t)\Big] \\
& = x\Big[y_t(x,t)y_{xt}(x,t)+(1-v^2)y_{xx}(x,t)y_x(x,t)+\frac{3}{2}by_x^3(x,t)y_{xx}(x,t)\Big] \\
& = x\Big[\frac{1}{2}\left(y_t^2(x,t)\right)_x+\frac{1-v^2}{2}\left(y_x^2(x,t)\right)_x+\frac{3}{8}b\left(y_x^4(x,t)\right)_x\Big]
\end{aligned}
$$

and integrating the right-hand side of Eq. (20) by parts, one has

$$
\begin{aligned}
\dot{V}(t)-\dot{E}(t)=\gamma\Big[& \frac{1}{2}y_t^2(1,t)+\frac{1-v^2}{2}y_x^2(1,t)+\frac{3}{8}by_x^4(1,t) \\
& -\frac{1}{2}\int_0^1\left(y_t^2(x,t)+(1-v^2)y_x^2(x,t)+\frac{3}{4}by_x^4(x,t)\right)\mathrm{d}x\Big],
\end{aligned}
$$

hence Eq. (15) is obtained by using Eq. (14). The lemma is thus proved. □

As an initial investigation, the stabilizing control input is proposed as follows:

$$
u(t)=-ky_t(1,t) \tag{21}
$$

for all $t\geqslant 0$, where k is a constant real number. Under this situation, the actual external control force is $F_c(t)=-(k+v)y_t(1,t)$.

Lemma 2. *The time derivative of the function $V(t)$ in Eq.* (9), *along system* (8) *and* (21) *satisfies*

$$\dot{V}(t) = -\frac{\gamma}{2}\int_0^1 \left[y_t^2(x,t) + \left(1-v^2\right)y_x^2(x,t) + \frac{3}{4}by_x^4(x,t)\right]\mathrm{d}x + G(t) \tag{22}$$

for all $t \geqslant 0$, where

$$\begin{aligned} G(t) &:= \frac{1-v^2}{2}\left[\gamma(1-v^2+k^2) - 2(v+k)(1-v^2)\right]\frac{y_t^2(1,t)}{\left(1-v^2+1/2by_x^2(1,t)\right)^2} \\ &\quad + \frac{b}{2}\left[\gamma(1-v^2+\frac{3}{4}k^2) - 2(v+k)(1-v^2)\right]\frac{y_t^2(1,t)y_x^2(1,t)}{\left(1-v^2+1/2by_x^2(1,t)\right)^2} \\ &\quad + \frac{b^2}{8}[\gamma - 2(v+k)]\frac{y_t^2(1,t)y_x^4(1,t)}{\left(1-v^2+\frac{1}{2}by_x^2(1,t)\right)^2}. \end{aligned} \tag{23}$$

Proof. By Lemma 1, it only needs to prove $F(t) = G(t)$.

Case 1: $k \neq 0$. In this case, the boundary condition at $x = 1$, Eq. (8b), becomes

$$\frac{y_x(1,t)}{k} = -\frac{y_t(1,t)}{1-v^2+\frac{1}{2}by_x^2(1,t)}, \tag{24}$$

that is,

$$y_t(1,t) = -\frac{1-v^2}{k}y_x(1,t) - \frac{b}{2k}y_x^3(1,t). \tag{25}$$

Substituting Eq. (25) into Eq. (16) yields

$$\begin{aligned} F(t) &:= \left(\frac{1}{2}\gamma - v\right)\left[-\frac{1-v^2}{k}y_x(1,t) - \frac{b}{2k}y_x^3(1,t)\right]^2 \\ &\quad + \left(1-v^2\right)y_x(1,t)\left[-\frac{1-v^2}{k}y_x(1,t) - \frac{b}{2k}y_x^3(1,t)\right] \\ &\quad + \frac{b}{2}y_x^3(1,t)\left[-\frac{1-v^2}{k}y_x(1,t) - \frac{b}{2k}y_x^3(1,t)\right] + \frac{1-v^2}{2}\gamma y_x^2(1,t) + \frac{3}{8}b\gamma y_x^4(1,t) \\ &= y_x^2(1,t)\left[\left(\frac{1}{2}\gamma - v\right)\frac{(1-v^2)^2}{k^2} - \frac{(1-v^2)^2}{k} + \frac{1-v^2}{2}\gamma\right] \\ &\quad + y_x^4(1,t)\left[\left(\frac{1}{2}\gamma - v\right)\frac{b(1-v^2)}{k^2} - \frac{b(1-v^2)}{k} + \frac{3}{8}b\gamma\right] \\ &\quad + y_x^6(1,t)\left[\left(\frac{1}{2}\gamma - v\right)\frac{b^2}{4k^2} - \frac{b^2}{4k}\right] \\ &= \frac{1-v^2}{2}\left[\gamma(1-v^2+k^2) - 2(v+k)(1-v^2)\right]\frac{y_x^2(1,t)}{k^2} \\ &\quad + \frac{b}{2}\left[\gamma\left(1-v^2+\frac{3}{4}k^2\right) - 2(v+k)(1-v^2)\right]\frac{y_x^2(1,t)}{k^2}y_x^2(1,t) \\ &\quad + \frac{b^2}{8}[\gamma - 2(v+k)]\frac{y_x^2(1,t)}{k^2}y_x^4(1,t), \end{aligned}$$

hence substituting Eq. (24) into the above equation yields

$$F(t) = \frac{1-v^2}{2}[\gamma(1-v^2+k^2) - 2(v+k)(1-v^2)]\frac{y_t^2(1,t)}{\left[1-v^2+\frac{1}{2}by_x^2(1,t)\right]^2} + \frac{b}{2}\left[\gamma(1-v^2+\frac{3}{4}k^2) - 2(v+k)(1-v^2)\right]\frac{y_t^2(1,t)y_x^2(1,t)}{\left[1-v^2+\frac{1}{2}by_x^2(1,t)\right]^2} + \frac{b^2}{8}[\gamma - 2(v+k)]\frac{y_t^2(1,t)y_x^4(1,t)}{\left[1-v^2+\frac{1}{2}by_x^2(1,t)\right]^2},$$

which is just $G(t)$.

Case 2: $k=0$. In this case, the boundary condition at $x=1$, Eq. (8b), becomes

$$\left[1-v^2+\frac{1}{2}by_x^2(1,t)\right]y_x(1,t) = 0, \tag{26}$$

that is,

$$y_x(1,t) = 0. \tag{27}$$

Substituting Eq. (27) into Eq. (16) follows

$$F(t) = \left(\frac{1}{2}\gamma - v\right)y_t^2(1,t),$$

while substituting $k = 0$ and Eq. (27) into Eq. (23) yields

$$G(t) = \frac{1-v^2}{2}\left[\gamma(1-v^2) - 2v(1-v^2)\right]\frac{y_t^2(1,t)}{(1-v^2)^2} = \left(\frac{1}{2}\gamma - v\right)y_t^2(1,t).$$

Therefore both cases imply $F(t) = G(t)$, the lemma is thus proved. □

Proposition 2. *Let k and γ satisfy*

$$k > -v, \quad 0 < \gamma < \frac{2(v+k)(1-v^2)}{1-v^2+k^2}. \tag{28}$$

Then, the time derivative of the function V(t) in Eq. (9), along systems (8) and (21) satisfies

$$\dot{V}(t) \leqslant -\gamma E(t) \tag{29}$$

for all $t \geqslant 0$.

Proof. From Eq (28), one has $v+k>0$, $0<\gamma$, and $\gamma(1-v^2+k^2) - 2(v+k)(1-v^2)<0$, which follow $\gamma(1-v^2+3/4k^2) - 2(v+k)(1-v^2)<0$ and $\gamma - 2(v+k)<0$, hence $G(t)<0$. Using Lemma 2, Proposition 2 is demonstrated. □

Therefore it is proved that the functions $V(t)$ and $E(t)$ tend to zero exponentially.

Theorem 1. *Let k and γ satisfy*

$$k > -v, \quad 0 < \gamma < \min\left\{1-v, \frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}. \tag{30}$$

Then, the functions $V(t)$ and $E(t)$, along the solution of systems (8) and (21) satisfy

$$0 \leqslant V(t) \leqslant V(0)e^{-\gamma t/K_2^*}, \quad 0 \leqslant E(t) \leqslant \frac{V(0)}{K_1^*}e^{-\gamma t/K_2^*} \tag{31}$$

for all $t \geqslant 0$, where K_1^* and K_2^* are given in Eq. (13).

Proof. By Propositions 1 and 2, one has

$$\dot{V}(t) \leqslant -\gamma E(t) \leqslant -\frac{\gamma}{K_2^*} V(t)$$

for all $t \geqslant 0$, which implies the first inequality of Eq. (31). By Eq. (12) and the first inequality of Eq. (31), it can be concluded that the second inequality of Eq. (31) holds. The theorem is thus proved. □

In order to compare the theorem with that presented by Shahruz in Ref. [16], a new lemma is suggested as follows.

Lemma 3. *Let*

$$K_1 = 1 - \frac{\gamma(1+2v)}{1-v^2}, \quad K_2 = 1 + \frac{\gamma(1+2v)}{1-v^2}. \tag{32}$$

If $\gamma > 0$, then

$$K_1 < K_1^*, \quad K_2^* < K_2. \tag{33}$$

If $k > 0$, then

$$\min\left\{\frac{1-v^2}{1+2v}, 2v, \frac{4(1-v^2)}{3k}\right\} < \min\left\{1-v, \frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}. \tag{34}$$

Proof. First

$$K_1^* - K_1 = \frac{\gamma v}{1-v^2}, \quad K_2 - K_2^* = \frac{\gamma v}{1-v^2}, \quad 1 - v - \frac{1-v^2}{1+2v} = \frac{1-v}{1+2v} v,$$

thus Eq. (33) is obtained and one has

$$\frac{1-v^2}{1+2v} < 1 - v. \tag{35}$$

Next it will be supposed that the inequality

$$\min\left\{2v, \frac{4(1-v^2)}{3k}\right\} < \frac{2(v+k)(1-v^2)}{1-v^2+k^2}$$

is false and a contradiction will be derived.

Suppose

$$\min\left\{2v, \frac{4\left(1-v^2\right)}{3k}\right\} \geqslant \frac{2(v+k)\left(1-v^2\right)}{1-v^2+k^2}$$

and denote

$$\gamma_0 = \frac{2(v+k)\left(1-v^2\right)}{1-v^2+k^2}.$$

Then $2v \geqslant \gamma_0, 2(1-v^2)/k > \gamma_0$, which imply

$$(2v - \gamma_0)(1-v^2) + \left[\frac{2\left(1-v^2\right)}{k} - \gamma_0\right] k^2 > 0. \tag{36}$$

While

$$(2v-\gamma_0)\left(1-v^2\right)+\left[\frac{2\left(1-v^2\right)}{k}-\gamma_0\right]k^2=(v+k)\left(1-v^2\right)-\gamma_0\left(1-v^2+k^2\right),$$

Eq. (36) implies $2(v+k)(1-v^2)/(1-v^2+k^2)>\gamma_0$. This contradicts the definition of γ_0. The contradiction proves the inequality

$$\min\left\{2v,\frac{4(1-v^2)}{3k}\right\}<\frac{2(v+k)(1-v^2)}{1-v^2+k^2} \tag{37}$$

Now let us prove Eq. (34). If

$$\min\left\{1-v,\frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}=1-v,$$

Eq. (35) implies

$$\min\left\{1-v,\frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}=1-v>\frac{1-v^2}{1+2v}\geqslant\min\left\{\frac{1-v^2}{1+2v},2v,\frac{4(1-v^2)}{3k}\right\}.$$

If

$$\min\left\{1-v,\frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}=\frac{2(v+k)(1-v^2)}{1-v^2+k^2},$$

then Eq. (37) implies

$$\begin{aligned}\min\left\{1-v,\frac{2(v+k)(1-v^2)}{1-v^2+k^2}\right\}&=\frac{2(v+k)(1-v^2)}{1-v^2+k^2}>\min\left\{2v,\frac{4(1-v^2)}{3k}\right\}\\&\geqslant\min\left\{\frac{1-v^2}{1+2v},2v,\frac{4(1-v^2)}{3k}\right\}.\end{aligned}$$

Therefore, one obtains Eq. (34). The lemma is thus proved. □

By Theorem 1 and Lemma 3, the following corollary holds, which was given in Ref. [16].

Corollary 1. *Let k and γ satisfy*

$$k>0,\quad 0<\gamma<\min\left\{\frac{1-v^2}{1+2v},2v,\frac{4(1-v^2)}{3k}\right\}. \tag{38}$$

Then, the functions V(t) and E(t), along the solution of systems (8) and (21) satisfy

$$0\leqslant V(t)\leqslant V(0)e^{-\gamma t/K_2},\quad 0\leqslant E(t)\leqslant\frac{V(0)}{K_1}e^{-\gamma t/K_2} \tag{39}$$

for all $t\geqslant 0$, where K_1 and K_2 are given in Eq. (32).

Finally, it is shown that the boundary control $u(t)$ in Eq. (21) stabilizes the nonlinear string in Eq. (8).

Theorem 2. *Let k satisfy*

$$k>-v. \tag{40}$$

Then, the solution $y(x,t)$ of systems (8) and (21) tends to zero exponentially as $t\to+\infty$ uniformly for all$x\in[0,1]$.

Proof. Using the fundamental theorem of calculus and the fact $y(0,t)=0$ for all $t\geqslant 0$, one obtains

$$|y(x,t)|=\left|\int_0^x y_\xi(\xi,t)\,\mathrm{d}\xi\right|\leqslant\int_0^x|y_\xi(\xi,t)|\,\mathrm{d}\xi\leqslant\int_0^1|y_\xi(\xi,t)|\,\mathrm{d}\xi\leqslant\left(\int_0^1 y_\xi^2(\xi,t)\,\mathrm{d}\xi\right)^{1/2}\leqslant\left(\frac{2E(t)}{1-v^2}\right)^{1/2}$$

for all $x \in [0, 1]$ and $t \geqslant 0$, where the last two inequalities follow from Cauchy–Schwarz inequality and Eq. (10). For systems (8) and (21), a γ is chosen such that the second inequality of Eq. (30) holds. By Theorem 1, the function $E(t)$ tends to zero exponentially as $t \to +\infty$, so $y(x, t)$ tends to zero exponentially as $t \to +\infty$, uniformly for all $x \in [0,1]$. The theorem is thus proved.

Recalling that $F_c(t) = -(k + v)y_t(1, t)$, it can be drawn from Theorem 2 that the transverse response of the moving string subject to finite deformation can be suppressed to exponentially stable only if the externally applied control force satisfies $F_c(t) = -k'y_t(1, t)$ and $k' > 0$.

4. Conclusions

In this paper, a nonlinear model of a moving string subject to boundary control is established by means of the Hamilton's principle. Lyapunov-type energy functionals are constructed, for which some lemmas, theorems and corollaries are presented and proved. Demonstrations show that the oscillation of a moving string with finite deformation is exponentially stable when a linear boundary control of negative speed feedback is applied.

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